

# Quantum probabilities for time-extended alternatives

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## Abstract

We study the probability assignment for the outcomes of time-extended measurements. We construct the class-operator that incorporates the information about a generic time-smeared quantity. These class-operators are employed for the construction of Positive-Operator-Valued-Measures for the time-averaged quantities. The scheme highlights the distinction between velocity and momentum in quantum theory. Propositions about velocity and momentum are represented by different class-operators, hence they define different probability measures. We provide some examples, we study the classical limit and we construct probabilities for generalized time-extended phase space variables.

## 1 Introduction

In this article we study the probability assignment for quantum measurements of observables that take place in finite time. Usually measurements are treated as instantaneous. One assumes that the duration of interaction between the measured system and the macroscopic measuring device is much smaller than any macroscopic time scale characterising the behaviour of the measurement device. Although this is a reasonable assumption, measurements that take place in a macroscopically distinguishable time interval are theoretically conceivable, too. In the latter case one expects that the corresponding probabilities would be substantially different from the ones predicted by the instantaneous approximation. Moreover, the consideration of the duration of the measurement as

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a determining parameter allows one to consider observables whose definition explicitly involves a finite time interval. Such observables may not have a natural counterpart when restricted to single-time alternatives. In what follows, we also study physical quantities whose definition involves time-derivatives of single-time observables.

There are different procedures we can follow for the study of finite-time measurements. For example, one may employ standard models of quantum measurement and refrain from taking the limit of almost instantaneous interaction between the measuring system and the apparatus [1]. However, there is an obvious drawback. For example, a measurement of momentum can be implemented by different models for the measuring device. They all give essentially a probability that is expressed in terms of momentum spectral projectors (more generally positive operators). However, if one considers a measurement of finite duration, it is not obvious to identify the physical quantity of the measured system to which the resulting probability measure corresponds.

This problem is especially pronounced when one considers measurements of relatively large duration. For the reason above, we choose a different starting point: we identify time-extended classical quantities and the we construct corresponding operators that act on the Hilbert space of the measured system. A special case of such observables are quantities that are smeared in time. If an operator  $\hat{A}$  has (generalised) eigenvalues  $a$ , then we identify a probability density for its time-smeared values  $\langle a \rangle_f = \int_0^T dt a_t f(t)$ . Here  $f(t)$  is a positive function defined on the interval  $[0, T]$ . The special case  $f(t) = \frac{1}{T}$  corresponds to the usual notion of time-averaging.

Having identified the operators that represent the time-extended quantities, it is easy to construct the corresponding probability measure for such observables using for example, simple models for quantum measurement.

Our analysis is facilitated by a comparison with the decoherent histories approach to quantum mechanics [2, 3, 4, 5]. The identification of operators that correspond to time-extended observables is structurally similar to the description of temporally extended alternatives in the decoherent histories approach [6, 7, 8, 9, 10, 11, 12]. The physical context is different, in the sense that the decoherent histories scheme attempts the description of individual closed systems, while the study of measurements we undertake here involves—by necessity—the consideration of open systems. However, the mathematical descriptions are very closely related.

A history is defined as a sequence of propositions about the physical system at successive moments of time. A proposition in quantum mechanics is represented by a projection operator; hence, a general  $n$ -time history  $\alpha$  corresponds to a string of projectors  $\{\hat{P}_{t_1}, \hat{P}_{t_2}, \dots, \hat{P}_{t_n}\}$ . To determine the probabilities associated to these histories we define the class operator  $\hat{C}_\alpha$ ,

$$\hat{C}_\alpha = \hat{U}^\dagger(t_1) \hat{P}_{t_1} \hat{U}(t_1) \dots \hat{U}^\dagger(t_n) \hat{P}_{t_n} \hat{U}(t_n), \quad (1)$$

where  $\hat{U}(t) = e^{-i\hat{H}t}$  is the evolution operator for the system. For a pair of histories  $\alpha$  and  $\alpha'$ , we define the decoherence functional

$$d(\alpha, \alpha') = \text{Tr} \left( \hat{C}_\alpha^\dagger \hat{\rho}_0 \hat{C}_{\alpha'} \right). \quad (2)$$

A key feature of the decoherent histories scheme is that probabilities can be assigned to an exclusive and exhaustive set of histories only if the decoherence condition

$$d(\alpha, \alpha') = 0, \quad \alpha \neq \alpha' \quad (3)$$

holds. In this case one may define a probability measure on this space of histories

$$p(\alpha) = \text{Tr} \left( \hat{C}_\alpha^\dagger \hat{\rho} \hat{C}_\alpha \right). \quad (4)$$

One of the most important features of the decoherent histories approach is its rich logical structure: logical operations between histories can be represented in terms of algebraic relations between the operators that represent a history. This logical structure is clearly manifested in the History Projection Operator (HPO) formulation of decoherent histories [13]. In this paper we will make use of the following property. If  $\{\alpha_i\}$  is a collection of mutually exclusive histories, each represented by the class operator  $\hat{C}_{\alpha_i}$  then the coarse-grained history that corresponds to the statement that any one of the histories  $i$  has been realised is represented by the class operator  $\sum_i \hat{C}_{\alpha_i}$ . This property has been employed by Bosse and Hartle [12], who define class operators corresponding to time-averaged position alternatives using path-integrals. A similar construction in a slightly different context is given by Sokolovski et al [14, 15]—see also Ref. [16].

Our first step is to generalise the results of [12] by constructing such class operators for the case of a generic self-adjoint operator  $\hat{A}$  that are smeared with an arbitrary function  $f(t)$  within a time interval  $[0, T]$ . This we undertake in section 2.

In section 3, we describe a toy model for a time-extended measurement. It leads to a probability density for the measured observable that is expressed solely in terms of the class operators  $\hat{C}_\alpha$ . The same result can be obtained without the use of models for the measurement device through a purely mathematical argument. We identify generic Positive-Operator-Valued Measure (POVM) that is bilinear with respect to the class operators  $\hat{C}_\alpha$  and compatible with Eq. (4).

The result above implies that  $\hat{C}_\alpha$  can be employed in two different roles: first, as ingredients of the decoherence functional in the decoherent histories approach and second, as building block of a POVM in an operational approach to quantum theory. The same mathematical object plays two different roles: in [12] class operators corresponding to time-average observables are constructed for use within the decoherent histories approach, while the same objects are used in [15] for the determination of probabilities of time-extended position measurements.

The approach we follow allows the definition of more general observables. Within the context of the HPO approach, velocity and momentum are represented by different (non-commuting) operators: they are in principle distinguishable concepts [9].

In section 4, we show that indeed one may assign class operators to alternatives corresponding to values of velocity that are distinct from those corresponding to values of momentum. These operators coincide at the limit of large coarse-graining (which often coincides with the classical limit). In effect, two quantities that coincide in classical physics are represented by different objects quantum mechanically. It is quite interesting that the POVMs corresponding to velocity are substantially different from those corresponding to momentum. At the formal level, it seems that quantum theory allows the existence of instruments which are able to distinguish between the velocity and momentum of a quantum particle. *A priori*, this is not surprising: in single-time measurements, velocity cannot be defined as an independent variable. For extended-in-time measurements, it is not inconceivable that one type of detector responds to the rate of change of the position variable and another to the particle’s momentum. Whether this result is a mere mathematical curiosity, or whether one can design experiments that will demonstrate this difference completely will be addressed in a future publication. In section 4 we also study more general time-extended measurements, namely ones that correspond to time-extended phase space properties of the quantum system.

## 2 Operators representing time-averaged quantities

### 2.1 The general form of the class operators

We construct the class operators that correspond to the proposition “the value of the observable  $\hat{A}$ , smeared with a function  $f(t)$  within a time interval  $[0, T]$ , takes values in the subset  $U$  of the real line  $\mathbf{R}$ .”

We denote by  $a_t$  a possible value of the observable  $\hat{A}$  at time  $t$ . Then at the continuous-time limit the time-smeared value  $A_f$  of  $\hat{A}$  reads  $A_f := \int_0^T a_t f(t) dt$ . Note that for the special choice  $f(t) = \frac{1}{T} \chi_{[0, T]}(t)$ , where  $\chi_{[0, T]}$  is the characteristic function of the interval  $[0, T]$ , we obtain the usual notion of the time-averaged value of a physical quantity.

There are two benefits from the introduction of a general function  $f(t)$ . First, it can be chosen to be a continuous function of  $t$ , thus allowing the consideration of more general ‘observables’; for example observables that involve the time derivatives of  $a_t$ . Second, when we consider measurements, the form of  $f(t)$  may be determined by the operations we effect on the quantum system. For example,  $f(t)$  may correspond to the shape of an electromagnetic pulse acting upon a charged particle during measurement.

To this end, we construct the relevant class operators in a discretised form. We partition the interval  $[0, T]$  into  $n$  equidistant time-steps  $t_1, t_2, \dots, t_n$ . The integral  $\int_0^T dt f(t) a_t$  is obtained as the continuous limit of  $\delta t \sum_i f(t_i) a_{t_i} = \frac{T}{n} \sum_i f(t_i) a_{t_i}$ .

For simplicity of exposition we assume that the operator  $\hat{A}$  has discrete spectrum, with eigenvectors  $|a_i\rangle$  and corresponding eigenvalues  $a_i$ <sup>1</sup>. We write  $\hat{P}_{a_i} = |a_i\rangle\langle a_i|$ . By virtue of Eq. (1) we construct the class operator

$$\hat{C}_\alpha = e^{i\hat{H}T/n} |a_1\rangle\langle a_1| e^{i\hat{H}T/n} |a_2\rangle\langle a_2| \dots |a_{n-1}\rangle e^{i\hat{H}T/n} |a_n\rangle\langle a_n| \quad (5)$$

that represents the history  $\alpha = (a_1, \dots, a_n)$ .

The proposition “the time-averaged value of  $\hat{A}$  lies in a subset  $U$  of the real line” can be expressed by summing over all operators of the form of Eq. (5), for which  $\frac{T}{n} \sum_i f(t_i) a_i \in U$ ,

$$\begin{aligned} \hat{C}_U = & \sum_{a_1, a_2, \dots, a_n} \chi_U \left( \frac{T}{n} \sum_i f(t_i) a_i \right) \\ & \times e^{i\hat{H}T/n} |a_1\rangle\langle a_1| e^{i\hat{H}T/n} |a_2\rangle\langle a_2| \dots |a_{n-1}\rangle e^{i\hat{H}T/n} |a_n\rangle\langle a_n|. \end{aligned} \quad (6)$$

If we partition the real axis of values of the time-averaged quantity  $A_f$  into mutually exclusive and exhaustive subsets  $U_i$ , the corresponding alternatives for the value of  $A_f$  will also be mutually exclusive and exhaustive.

Next, we insert the Fourier transform  $\tilde{\chi}_U$  of  $\chi_U$  defined by

$$\chi_U(x) := \int \frac{dk}{2\pi} e^{ikx} \tilde{\chi}_U(k) \quad (7)$$

into Eq. (6). We thus obtain

$$\begin{aligned} \hat{C}_U = & \int \frac{dk}{2\pi} \tilde{\chi}_U(k) e^{-i\hat{H}T/n} \left( \sum_{a_1} e^{-ikTf(t_1)a_1/n} |a_1\rangle\langle a_1| \right) e^{i\hat{H}T/n} \dots \\ & \times e^{i\hat{H}T/n} \left( \sum_{a_n} e^{-ikTf(t_n)a_n/n} |a_n\rangle\langle a_n| \right). \end{aligned} \quad (8)$$

By virtue of the spectral theorem we have

$$\sum_{a_i} e^{ikTf(t_i)a_i/n} |a_i\rangle\langle a_i| = e^{ikf(t_i)\hat{A}/n}. \quad (9)$$

Hence,

$$\hat{C}_U = \int \frac{dk}{2\pi} \tilde{\chi}_U(k) \prod_{i=1}^n [e^{i\hat{H}T/n} e^{ikf(t_i)\hat{A}T/n}]. \quad (10)$$

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<sup>1</sup>The generalization of our results for continuous spectrum is straightforward.

From Eq. (10) we obtain

$$\hat{C}_U = \int_U da \hat{C}(a), \quad (11)$$

where

$$\hat{C}(a) := \int \frac{dk}{2\pi} e^{-ika} \hat{U}_f(T, k), \quad (12)$$

and where

$$\hat{U}_f(T, k) := \lim_{n \rightarrow \infty} \prod_{i=1}^n [e^{i\hat{H}T/n} e^{-ikf(t_i)\hat{A}T/n}]. \quad (13)$$

The operator  $\hat{U}_f$  is the generator of an one-parameter family of transformations

$$-i \frac{\partial}{\partial s} \hat{U}_f(s, k) = [\hat{H} + kf(s)\hat{A}] \hat{U}_f(s, k). \quad (14)$$

This implies that

$$\hat{U}_f(T, k) = \mathcal{T} e^{i \int_0^T dt (H + kf(t)\hat{A})}, \quad (15)$$

where  $\mathcal{T}$  signifies the time-ordered expansion for the exponential. The construction of  $\hat{C}_U$  then is mathematically identical to the determination of a propagator in presence of a time-dependent external force proportional to  $\hat{A}$ .

For  $f(t) = \frac{1}{T}\chi_{[0,T]}(t)$  we obtain

$$\hat{C}_U = \int \frac{dk}{2\pi} \tilde{\chi}_U(k) e^{i\hat{H}T + ik\hat{A}}, \quad (16)$$

that has been constructed through path-integrals for specific choices of the operator  $\hat{A}$  in [14, 15, 12].

If  $f(t)$  has support in the interval  $[t, t'] \subset [0, T]$  then

$$\hat{C}_U = e^{-i\hat{H}t} \int_U da \left( \int \frac{dk}{2\pi} e^{-ika} \mathcal{T} e^{i \int_t^{t'} ds (H + kf(s)\hat{A})} \right) e^{i\hat{H}(T-t')}. \quad (17)$$

We note that outside the interval  $[t, t']$  only the Hamiltonian evolution contributes to  $\hat{C}_U$  outside the interval  $[t, t']$ .

It will be convenient to represent the proposition about the time-averaged value of  $\hat{A}$  by the operator

$$\hat{D}(a) := e^{-i\hat{H}T} \hat{C}(a), \quad (18)$$

or else

$$\hat{D}(a) = \int \frac{dk}{2\pi} e^{-ika} \mathcal{T} e^{ik \int_0^T dt f(t)\hat{A}(t)}, \quad (19)$$

where  $\hat{A}(t)$  is the Heisenberg-picture operator  $e^{i\hat{H}t}\hat{A}e^{-i\hat{H}t}$ .

If  $[\hat{H}, \hat{A}] = 0$ , then

$$\hat{U}_f(T, k) = e^{i\hat{A} \int_0^T dt f(t)}. \quad (20)$$

Hence,

$$\hat{D}_U := \int_U da \hat{D}(a) = \chi_U [\hat{A} \int_0^T dt f(t)]. \quad (21)$$

When we use  $f(t)$  to represent time-smearing, it is convenient to require that  $\int_0^T dt f(t) = 1$  in order to avoid any rescaling in the values of the observable. Then  $\hat{D}_U = \chi_U(\hat{A})$ . We conclude therefore that the operator representing time-averaged value of  $\hat{A}$  coincides with the one representing a single-time value of  $\hat{A}$ .

**The limit of large coarse-graining.** If we integrate  $\hat{D}(a)$  over a relatively large sample set  $U$  the integral over  $dk$  is dominated by small values of  $k$ . To see this, we approximate the integration over a subset of the real line of width  $\Delta$  centered around  $a = a_0$ , by an integral with a smeared characteristic function  $\exp[-(a - a_0)^2/2\Delta^2]$ . This leads to

$$\hat{D}_U = \sqrt{2\pi}\Delta \int \frac{dk}{2\pi} e^{-\Delta^2 k^2/2} \mathcal{T} e^{i \int_0^T dt f(t) \hat{A}(t)} \quad (22)$$

that is dominated by values of  $k \sim \Delta^{-1}$ .

The term  $k f(t)$  in the time-ordered exponential of Eq. (19) is structurally similar to a coupling constant. Hence, for sufficiently large values of  $\Delta$  we write

$$\mathcal{T} e^{i \int_0^T dt f(t) \hat{A}(t)} \simeq e^{i \int_0^T dt f(t) \hat{A}(t)}, \quad (23)$$

i.e., the zero-th loop order contribution to the time-ordered exponential dominates. We therefore conclude that

$$\hat{D}_U \simeq \chi_U \left[ \int_0^T dt f(t) \hat{A}(t) \right]. \quad (24)$$

$\hat{D}_U$  is almost equal to a spectral element of the time-averaged Heisenberg-picture operator  $\int_0^T dt f(t) \hat{A}(t)$ . This generalises the result of [12], which was obtained for configuration space variables at the limit  $\hbar \rightarrow 0$ .

We estimate the leading order correction to the approximation involved in Eq. (24). The immediately larger contribution to the time-ordered exponential of Eq. (19) is

$$\frac{k^2}{2} \int_0^T ds \int_0^s ds' f(s) f(s') [\hat{A}(s), \hat{A}(s')]. \quad (25)$$

The contribution of this term must be much smaller than the first term in the expansion of the time-ordered exponential's, namely  $k \int_0^T ds f(s) \hat{A}(s)$ . We write the expectation values of these operators on a vector  $|\psi\rangle$  in order to obtain the following condition

$$\left| \int_0^T ds \int_0^s ds' f(s) f(s') \langle \psi | [\hat{A}(s), \hat{A}(s')] | \psi \rangle \right| << \Delta \left| \int_0^T ds \langle \psi | \hat{A}(s) | \psi \rangle \right|. \quad (26)$$

The above condition is satisfied rather trivially for bounded operators if  $\|\hat{A}\| << \Delta$ . In that case, the operator  $\hat{C}_U$  captures little, if anything, from the possible values of  $\hat{A}$ . In the generic case however, Eq. (26) is to be interpreted as a *condition* on the state  $|\psi\rangle$ . Eq. (24) provides a good approximation if the two-time correlation functions of the system are relatively small.

Furthermore, if the function  $f(t)$  corresponds to weighted averaging, i.e., if  $f(t) \geq 0$ , and if  $f$  does not have any sharp peaks, then the condition  $\int_0^T dt f(t) = 1$  implies that the values of  $f(t)$  are of the order  $\frac{1}{T}$ .

We denote by  $\tau$  the correlation time of  $\hat{A}(s)$ , i.e. the values of  $|s - s'|$  for which  $|\langle \psi | [\hat{A}(s), \hat{A}(s')] | \psi \rangle|$  is appreciably larger than zero. Then at the limit  $T \gg \tau$  the left-hand side of Eq. (26) is of the order  $O\left(\frac{\tau^2}{T^2}\right)$ . Hence, for sufficiently large values of  $T$  one expects that Eq. (24) will be satisfied with a fair degree of accuracy.

The argument above does not hold if  $f$  is allowed to take on negative values, which is the case for the velocity samplings that we consider in section 4.

## 2.2 Examples

We study some interesting examples of class operators corresponding to time-smeared quantities. In particular, we consider the time-smeared position for a particle and a simple system that is described by a finite-dimensional Hilbert space.

### 2.2.1 Two-level system

In a two-level system described by the Hamiltonian  $\hat{H} = \omega \hat{\sigma}_z$ , we consider time-averaged samplings of the values of the operator  $\hat{A} = \hat{\sigma}_x$ . We compute

$$\hat{U}(k, T) = \cos \sqrt{k^2 + \omega^2 T^2} \hat{1} + i \frac{\sin \sqrt{k^2 + \omega^2 T^2}}{\sqrt{k^2 + \omega^2 T^2}} (k \hat{\sigma}_x + \omega T \hat{\sigma}_z). \quad (27)$$

Then the class operator  $\hat{C}(a)$  is

$$\begin{aligned} \hat{C}(a) = & \frac{\omega T}{2\sqrt{1-a^2}} J_1(\omega T \sqrt{1-a^2}) \hat{1} + \frac{a\omega T}{2\sqrt{1-a^2}} J_1(\omega T \sqrt{1-a^2}) \hat{\sigma}_x \\ & + \frac{i\omega T}{2} J_0(\omega T \sqrt{1-a^2}) \hat{\sigma}_z, \end{aligned} \quad (28)$$

where  $J_n$  stands for the Bessel function of order  $n$ . Note that the expression above holds for  $|a| \leq 1$ . For  $|a| > 1$ ,  $\hat{C}(a) = 0$ , as is expected by the fact that  $\|\hat{\sigma}_x\| = 1$ .

### 2.2.2 Position samplings

The case  $\hat{A} = \hat{x}$  for ordinary time-averaging ( $f(t) = \frac{1}{T}$ ) has been studied in [15, 12] using path integral techniques. Here we generalise these results by considering the case of a general smearing function  $f(t)$ .

We consider the case of a harmonic oscillator of mass  $m$  and frequency  $\omega$ . The determination of the propagator  $\hat{U}_f(T, k)$  for a harmonic oscillator acted by an external time-dependent force is well-known. It leads to the following expression for the operator  $\hat{D}(a)$

$$\langle x | \hat{D}(a) | x' \rangle = \frac{m\omega}{2\pi B_f \sin \omega T} \exp \left[ \frac{-im\omega}{2 \sin \omega T} (\cos \omega T (x'^2 - x^2) - 2xx') + \frac{2}{B_f} (A_f x' + a)(x' - x) - \frac{2\omega C_f}{B_f^2 \sin \omega T} (x - x')^2 \right], \quad (29)$$

where

$$A_f := \frac{1}{\sin \omega T} \int_0^T ds \sin \omega s f(s) \quad (30)$$

$$B_f := \frac{1}{\sin \omega T} \int_0^T ds \sin \omega(T-s) f(s) \quad (31)$$

$$C_f := \frac{1}{\omega \sin \omega T} \int_0^T ds \sin \omega(T-s) f(s) \int_0^s ds' \sin \omega s' f(s'). \quad (32)$$

The corresponding operators for the free particle is obtained at the limit  $\omega \rightarrow 0$

$$\langle x | \hat{D}(a) | x' \rangle = \frac{m}{2\pi B_f T} \exp \left[ \frac{-im}{2T} \left( (x'^2 - x^2) + \frac{2}{B_f} (A_f x' - a)(x' - x) - \frac{2C_f}{B_f^2 T} (x' - x)^2 \right) \right], \quad (33)$$

where

$$A_f = \frac{1}{T} \int_0^T ds s f(s) \quad (34)$$

$$B_f = \frac{1}{T} \int_0^T ds (T-s) f(s) \quad (35)$$

$$C_f = \frac{1}{T} \int_0^T ds (T-s) f(s) \int_0^s ds' s' f(s'). \quad (36)$$

### 3 Probability assignment

#### 3.1 The decoherence functional

For a pair of histories  $(U, U')$  that correspond to different samplings of the time-smeared values of  $\hat{A}$  the decoherence functional  $d(U, U')$  is

$$d(U, U') = \text{Tr} \left( \hat{D}_U^\dagger e^{-i\hat{H}T} \hat{\rho}_0 e^{i\hat{H}T} \hat{D}_{U'} \right). \quad (37)$$

From the expression above, we can read the probabilities that are associated to any set of alternatives that satisfies the decoherence condition. In section 2, we established that in the limit of large coarse-graining, or for very large values of time  $T$ , the operators  $\hat{D}_U$  approximate projection operators. Hence, if we partition the real line of values of  $A_f$  into sufficiently large exclusive sets  $U_i$  the decoherence condition will be satisfied. A probability measure will be therefore defined as

$$p(U_i) = \text{Tr} \left[ \chi_{U_i} \left( \int_0^T dt f(t) \hat{A}(t) \right) e^{-i\hat{H}T} \hat{\rho}_0 e^{i\hat{H}T} \right]. \quad (38)$$

This is the same as in the case of a single-time measurement of the observable  $\int_0^T dt f(t) \hat{A}(t)$  taking place at time  $t = T$ . For further discussion, see [12].

#### 3.2 Probabilities for measurement outcomes

Next, we show that the class operators  $\hat{C}(a)$  can be employed in order to define a POVM for a measurement with finite duration. For this purpose, we consider a simple measurement scheme. We assume that the system interacts with a measurement device characterised by a continuous pointer basis  $|x\rangle$ . For simplicity, we assume that the self-dynamics of the measurement device is negligible. The interaction between the measured system and the apparatus is described by a Hamiltonian of the form

$$\hat{H}_{int} = f(t) \hat{A} \otimes \hat{K}, \quad (39)$$

where  $\hat{K}$  is the ‘conjugate momentum’ of the pointer variable  $\hat{x}$

$$\hat{K} = \int dk k |k\rangle\langle k|, \quad (40)$$

where  $\langle x|k\rangle = \frac{1}{\sqrt{2\pi}} e^{-ikx}$ . The initial state of the apparatus (at  $t = 0$ ) is assumed to be  $|\Psi_0\rangle$  and the initial state of the system corresponds to a density matrix  $\hat{\rho}_0$ .

With the above assumptions, the reduced density matrix of the apparatus at time  $T$  is

$$\hat{\rho}_{app}(T) = \int dk \int dk' \text{Tr} \left( \hat{U}_f^\dagger(T, k) \hat{\rho}_0 \hat{U}_f(T, k') \right) \langle k|\Psi_0\rangle\langle\Psi_0|k'\rangle |k\rangle\langle k'|, \quad (41)$$

where  $\hat{U}_f(T, k)$  is given by Eq. (15). Then, the probability distribution over the pointer variable  $x$  (after reduction) is

$$\langle x | \hat{\rho}_{app}(T) | x \rangle = \int \frac{dk dk'}{2\pi} e^{-i(k-k')x} \langle k | \Psi_0 \rangle \langle \Psi_0 | k' \rangle \text{Tr} \left( \hat{U}_f^\dagger(T, k) \hat{\rho}_0 \hat{U}_f(T, k') \right). \quad (42)$$

The probability that the pointer variable takes values within a set  $U$  is

$$p(U) = \text{tr} \left( e^{-i\hat{H}T} \hat{\rho}_0 e^{i\hat{H}T} \hat{\Pi}_U \right), \quad (43)$$

where

$$\hat{\Pi}_U = \int_U dx \hat{D}(w_x^*) \hat{D}^\dagger(w_x) := \int_U dx \hat{\Pi}_x, \quad (44)$$

where  $w_x(a) := \langle x - a | \Psi_0 \rangle$  and where we employed the notation

$$\hat{D}(w_x) = \int da w_x(a) \hat{D}(a), \quad (45)$$

The operators  $\hat{\Pi}_U$  define a POVM for the time-extended measurement of  $\hat{A}$ : they are positive by construction, they satisfy the property  $\hat{\Pi}_{U_1 \cup U_2} = \hat{\Pi}_{U_1} + \hat{\Pi}_{U_2}$ , for  $U_1 \cap U_2 = \emptyset$  and they are normalised to unity

$$\hat{\Pi}_{\mathbf{R}} = \int_{\mathbf{R}} dx \hat{\Pi}_x = 1. \quad (46)$$

Note that the smearing of the class-operators is due to the spread of the wave function of the pointer variable.

In what follows we employ for convenience a Gaussian function

$$w(a) = \frac{1}{(2\pi\delta^2)^{1/4}} e^{-\frac{a^2}{4\delta^2}}. \quad (47)$$

In the free-particle case, the class operators in Eq. (33) lead to the following POVM

$$\langle y | \hat{\Pi}_x | y' \rangle = \frac{m}{\sqrt{2\pi} A_f T} \exp \left[ - \left( \frac{m^2 \delta^2}{2A_f^2 T^2} + \frac{A_f^2}{8\delta^2} \left( 1 - \frac{2C_f}{A_f^2 T} \right)^2 \right) (y - y')^2 + \frac{im}{A_f T} x(y' - y) \right]. \quad (48)$$

In Eq. (48), we chose an even time-averaging function, i.e.  $f(s) = f(T - s)$ , in which case  $A_f = B_f$ .

The POVM in Eq. (44) may also be constructed without reference to a specific model for the measurement device. In particular, we partition the space of values for  $A_f$  into sets of width  $\delta$  and employ the expression Eq. (4) for the ensuing probabilities. It is easy to show that these probabilities are reproduced—up

to terms of order  $O(\delta)$ —by a POVM of the form Eq. (44), with the smearing function  $w$  of Eq. (47)<sup>2</sup>.

If we restrict our considerations to the above measurement model, then there is no way we can interpret the POVM of Eq. (44) as corresponding to values of  $A_f$ . This interpretation is possible by the explicit construction and by the identification (see Sec. 2) of the class operators  $\hat{C}(a)$  as the only mathematical objects that correspond to such time-averaged alternatives.

## 4 More general samplings

### 4.1 Velocity Vs momentum

Within the context of the History Projection Operator scheme, Savvidou showed that histories of momentum differ in general from histories of velocity, in the sense that they are represented by different mathematical objects [9]. The corresponding probabilities are also expected to be different. In single-time quantum theory the notion of velocity (that involves differentiation with respect to time) cannot be distinguished from the notion of momentum. However, when we deal with histories, time differentiation is defined *independently* of the evolution laws. One may therefore consider alternatives corresponding to different values of velocity.

In particular, if  $x_f = \int_0^T dt x_t f(t)$  denotes the time-smeared value of the position variable, we define the time-smeared value of the corresponding velocity variable as

$$\dot{x}_f := -x_{\dot{f}}, \quad (49)$$

provided that the function  $f$  satisfies  $f(0) = f(T) = 0$ .

Notice here that when we measure the time-averaged value of an observable within a time-interval  $[0, T]$ , we employ positive functions  $f(t)$  that are  $\cap$ -shaped and that they satisfy  $\int_0^T dt f(t) = 1$ . Such functions correspond to the intuitive notions of averaging the value of a quantity with a specific weight.

However, to determine the time-average velocity—weighted by a positive and normalised function  $f$ —one has to smear the corresponding position variable with the function  $\dot{f}(t)$  that in the general case is neither positive nor normalised. Therefore *the form of the smearing function determines the physical interpretation of the observable we consider* [18].

Next, we compare the class operators corresponding to the average of velocity and of momentum, with a common weight  $f$ . We denote the velocity class operator as

$$\hat{D}^{\dot{x}}(a) = \int \frac{dk}{2\pi} e^{-ika} \mathcal{T} e^{i \int_0^T dt \dot{f}(t) \hat{x}(t)}, \quad (50)$$

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<sup>2</sup>The proof follows closely an analogous one in [17]).

and the momentum class operators as

$$\hat{D}^p(a) = \int \frac{dk}{2\pi} e^{-ika} \mathcal{T} e^{i \int_0^T dt f(t) \hat{p}(t)}. \quad (51)$$

At the limit of large coarse-graining, the operator  $\hat{D}_U^{\dot{x}} := \int_U da \hat{D}^{\dot{x}}(a)$  is approximately equal to

$$\hat{D}_U^{\dot{x}} = \chi_U \left( \int_0^T dt \dot{f}(t) \hat{x}(t) \right) = \chi_U \left( \frac{1}{m} \int_0^T dt f(t) \hat{p}(t) \right), \quad (52)$$

i.e., the class-operator for time-averaged momentum coincides with that for time-averaged velocity. This result reproduces the classical notion that  $p = m\dot{x}$ . However, the limit of large coarse-graining may be completely trivial if the temporal correlations of position are large.

For the case of a free particle, with the convenient choice  $f(t) = \frac{\pi}{T} \sin \frac{\pi t}{T}$ , we obtain

$$\hat{D}_U^p = \int_U dp |p\rangle\langle p|, \quad (53)$$

$$\hat{D}_U^{\dot{x}} = \int_U da \left( \sqrt{\frac{4imT}{\pi^3}} \int dp e^{i\frac{4mT}{\pi^2}(a-p/m)^2} |p\rangle\langle p| \right). \quad (54)$$

It is clear that the alternatives of time-averaged momentum are distinct from those of time-averaged velocity. Still, at the limit  $T \rightarrow \infty$ ,  $\hat{D}_U^p = m\hat{D}_U^{\dot{x}}$ .

The POVM corresponding to Eq. (54) is

$$\hat{\Pi}^{\dot{x}}(v) = \frac{1}{\sqrt{2\pi\sigma_v^2(T)}} \int dp \exp \left[ -\frac{1}{2\sigma_v^2(T)} (v - p/m)^2 \right] |p\rangle\langle p|, \quad (55)$$

where  $\sigma_v^2(T) = \delta^2 + \frac{\pi^4}{2^8 m^2 T^2 \delta^2}$ .

The POVM of Eq. (55) commutes with the momentum operator. One could therefore claim that it corresponds to an unsharp measurement of momentum. However, the commutativity of this POVM with momentum follows only from the special symmetry of the Hamiltonian for a free-particle, it does not hold in general. Moreover, at the limit of small  $T$ , the distribution corresponding to Eq. (55) has a very large mean deviation. Hence, even for a wave-packet narrowly concentrated in momentum, the spread in measured values is large. Note that at the limit  $T \rightarrow 0$ , the deviation  $\sigma_v^2(T) \rightarrow \infty$  and the POVM (55) tends weakly to zero. For  $T \gg (m\delta^2)^{-1}$ , then  $\sigma_v^2(T) \simeq \delta^2$  and the velocity POVM is identical to one obtained by an instantaneous momentum measurement.

The results of section 3.2 suggest the different measurement schemes that are needed for the distinction of velocity and momentum. For a momentum measurement the interaction Hamiltonian should be of the form

$$\hat{H}_{int}^p = f(t) \hat{p} \otimes \hat{K}, \quad (56)$$

where  $f(t)$  is a  $\cap$ -shaped positive-valued function. For a velocity measurement the interaction Hamiltonian is

$$\hat{H}_{int}^x = -\dot{f}(t) \hat{x} \otimes \hat{K}. \quad (57)$$

The two Hamiltonians differ not only on the coupling but also on the shape of the corresponding smearing functions:  $\dot{f}(t)$  takes both positive and negative values and by definition it satisfies  $\int_0^T \dot{f}(t) = 0$ . The description above suggests that momentum measurements can be obtained by coupling a charged particle to a magnetic field pulse, while velocity measurements can be obtained by a coupling to an electric field pulse of a different shape. The possibility of designing realistic experiments that could distinguish between the momentum and the velocity content of a quantum state will be discussed elsewhere.

## 4.2 Lagrangian action

One may also consider samplings corresponding to the values of the Lagrangian action of the system  $\int_0^T dt L(x, \dot{x})$ , where  $L$  is the Lagrangian. In this case the results can be easily expressed in terms of Feynman path integrals: it is straightforward to demonstrate—see Ref. [14]—that these coincide with samplings of the Hamiltonian, and that the corresponding POVM is that of energy measurements.

## 4.3 Phase space properties

It is possible to construct class-operators (and corresponding POVMs) for more general alternatives that involve phase-space variables. To see this, we consider a set of coherent states  $|z\rangle$  on the Hilbert space, where  $z$  denotes points of the corresponding classical phase space. The finest-grained histories corresponding to an  $n$ -time coherent state path  $z_0, t_0, z_1, t_1, \dots, z_n, t_n$ , with  $t_i - t_{i-1} = \delta t$  are represented by the class operator

$$\hat{C}_{z_0, t_0; z_1, t_1; \dots; z_n, t_n} = |z_0\rangle\langle z_0| e^{i\hat{H}\delta t} |z_1\rangle\langle z_1| e^{i\hat{H}\delta t} |z_2\rangle\langle z_2| \dots |z_{n-1}\rangle\langle z_{n-1}| e^{i\hat{H}\delta t} |z_n\rangle\langle z_n|. \quad (58)$$

We use the standard Gaussian coherent states, which are defined through an inner product

$$\langle z|z'\rangle = e^{-\frac{|z|^2}{2} - \frac{|z'|^2}{2} + z^* z'}. \quad (59)$$

Then, at the limit of small  $\delta t$

$$\hat{C}_{z_1, t_1; z_2, t_2; \dots; z_n, t_n} = |z_0\rangle\langle z_n| \exp\left(\frac{|z_n|^2}{2} - \frac{|z_0|^2}{2} - \sum_{i=1}^n z_i^*(z_i - z_{i-1}) + i\delta t h(z_i^*, z_{i-1})\right), \quad (60)$$

where  $h(z^*, z) = \langle z|\hat{H}|z\rangle$ . Following the same steps as in section 2.1 we construct the class operator corresponding to different values of an observable

$A(z_0, z_1, \dots, z_n)$ . If the observable is ultra-local, i.e., if it can be written in the form  $\sum_i f(t_i) a(z_i)$ , then the results reduce to those of section 2.1 for the time-smeared alternatives of an operator.

However, the function in question may involve time derivatives of phase space variables (at the continuous limit), in which case it will be rather different from the ones we considered previously. For a generic function  $F(z_i)$  we obtain the following class operator that corresponds to the value  $F = a$

$$\begin{aligned} \langle z_0 | \hat{C}(a) | z_f \rangle &= \int \frac{dk}{2\pi} e^{-ika} \lim_{n \rightarrow \infty} \int [dz_1] \dots [dz_{n-1}] \\ &\times \exp \left[ \frac{|z_n|^2}{2} - \frac{|z_0|^2}{2} + \sum_i z_i (z_i^* - z_{i-1}^*) + i\delta t h(z_i^*, z_{i-1}) - ikF[z_i] \right]. \end{aligned} \quad (61)$$

The integrations over  $[dz_i]$  defines a coherent-state path-integral at the continuous limit. However, if  $F[z_i]$  is not an ultra-local function, the path integral does not correspond to a unitary operator of the form  $\mathcal{T}e^{\int_0^T dt \hat{K}_t}$ , for some family of self adjoint operators  $\hat{K}_t$ . In this sense, the consideration of phase space paths provides alternatives that do not reduce to those studied in Section 2. Note however, that these alternatives cannot be defined in terms of projection operators; nonetheless the corresponding class operators can be employed to define a POVM using Eq. (44).

The simplest non-trivial example of a non-ultralocal function is the Liouville term of the phase space action (for its physical interpretation in the histories theory see [9])

$$V := i \int_0^T dt \dot{z}^* z. \quad (62)$$

It is convenient to employ the discretised expression  $V = i \sum_{i=1}^n z_i (z_i^* - z_{i-1}^*)$ . Its substitution in Eq. (61) effects a multiplication of the Liouville term in the exponential by a factor of  $1 + k$ .

For an harmonic oscillator Hamiltonian  $h(z^*, z) = \omega z^* z$ , and the path integral can be explicitly computed yielding the unitary operator  $e^{\frac{i}{1+k} \hat{H}T}$ . Hence,

$$\hat{C}(a) = \int \frac{dk}{2\pi} e^{-ika} e^{\frac{i}{1+k} \hat{H}T} = s_a(\hat{H}), \quad (63)$$

where  $s_a(x) := \int \frac{dk}{2\pi} e^{-ika + i \frac{x}{1+k}}$ . The class-operator  $\hat{C}(a)$  corresponding to the values of the function  $V$  is then a function of the Hamiltonian.

## 5 Conclusions

We studied the probability assignment for time-extended measurements. We constructed of the class operators  $\hat{C}(a)$ , which correspond to time-extended

alternatives for a quantum system. We showed that these operators can be employed to construct POVMs describing the probabilities for time-averaged values of a physical quantity. In light of these results, quantum mechanics has room for measurement schemes that distinguish between momentum and velocity. Finally, we demonstrated that a large class of time-extended phase space observables may be explicitly constructed.

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